

Analog of the Peter-Weyl Expansion for Lorentz Group

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Dedicated to my teacher, Alexander Zibrov.

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Abstract

The expansion of a square integrable function on $SL(2, C)$ into the sum of the principal series matrix coefficients with the specially selected representation parameters was recently used in the Loop Quantum Gravity [10], [11]. In this paper we prove that the sum used originally in the Loop Quantum Gravity: $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm, jn}^{(j, \tau j)}(g)$, where $j, m, n \in Z, \tau \in C$ is convergent to a function on $SL(2, C)$, however the limit is not a square integrable function therefore such sums can not be used for the Peter-Weyl like expansion. We propose the alternative expansion and prove that for each fixed m : $\sum_{j=m}^{\infty} D_{jm, jm}^{(j, \tau j)}(g)$ is convergent and that the limit is a square integrable function on $SL(2, C)$. We then prove the analog of the Peter-Weyl expansion: any $\psi(g) \in L_2(SL(2, C))$ can be decomposed into the sum:
$$\psi(g) = \sum_{j=m}^{\infty} j^2 (1 + \tau^2) c_{jmm} D_{jm, jm}^{(j, \tau j)}(g),$$
 with the Fourier coefficients $c_{jmm} = \int_{SL(2, C)} \psi(g) \overline{D_{jm, jm}^{j, \tau j}(g)} dg, g \in SL(2, C), \tau \in C, \tau \neq i, -i, j, m \in Z, m$ is fixed. We also prove convergence of the sums $\sum_{j=|p|}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} d_{pm}^{\frac{j}{2}} D_{jm, jn}^{(j, \tau j)}(g)$, where $d_{pm}^{\frac{j}{2}} = (j+1)^{\frac{1}{2}} \int_{SU(2)} \phi(u) \overline{D_{pm}^{\frac{j}{2}}(u)} du$ is $\phi(u)$'s Fourier transform and $p, j, m, n \in Z, \tau \in C, u \in SU(2), g \in SL(2, C)$, thus establishing the map between the square integrable functions on $SU(2)$ and the space of the functions on $SL(2, C)$. Such maps were first used in [10].

1 Introduction

In this paper we show that a square integrable function on $SL(2, C)$ can be expanded into the sum of the principal series matrix coefficients with the parameters $D_{jm, j'n}^{(k, \rho)}$, for $k = j, \rho = j\tau, j = j', m = n, j \in Z, \tau \in C, m, n \in Z$, i.e. into the matrix

coefficients of the form $D_{jm,jm}^{(j,j\tau)}$ for the fixed m . While the Peter-Weyl theorem is applicable only to the compact groups, this decomposition is the analog of the Peter-Weyl expansion for the non-compact Lorentz group double cover $SL(2, C)$. Such specific selection of the principal series coefficient parameters is not accidental. In fact $k = j, \rho = j\tau$ is the simplicity constraints solution. The simplicity constraints, introduced by John Barrett and Louse Crane in [6] allow to consider the Quantum Gravity as a 4-dimensional topological model called BF-model plus some constraints on the form of the bivectors used in BF model. Those constraints are called the simplicity constraints. The simplicity constraints is what makes the 4-dim topological model to become Einstein's Quantum Gravity. Thus the principal series matrix coefficients of the form $D_{jm,jm}^{(j,j\tau)}$ have a special physical meaning. For the details please see [6], [7], [8] and [9].

In this paper the sum convergence proof is the main and the most challenging task. To prove convergence we use the following: a) the principal series matrix coefficients expression via the hypergeometric functions [1], b) formula (4.11), the Watson's asymptotic of the hypergeometric functions ${}_2F_1(a, b, c, z)$, when all three parameters tend to infinity [5], and c) the D'Alembert-Cauchy convergence ratio test.

The paper is organized as follows. In the next section 2 we prove convergence of the sums $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,j\tau)}$ and $\sum_{j=m}^{\infty} D_{jm,jm}^{(j,j\tau)}$. In section 3 we prove that while the limit of the first sum is not a square integrable function, the limit of the second sum is. We then prove that any square integrable function on $SL(2, C)$ $\psi(g)$ can be expanded into the sum $\psi(g) = \sum_{j=m}^{\infty} j^2(1 + \tau^2) c_{jmm} D_{jm,jm}^{(j,\tau j)}(g)$, with the $SL(2, C)$ Fourier coefficients $c_{jmm} = \int_{SL(2,C)} \psi(g) \overline{D_{jm,jm}^{j,\tau j}(g)} dg$ and $\tau \in C, \tau \neq i, -i$. In the section 4 we establish the map from the space of the square integrable functions on $SU(2)$ to the space of functions on $SL(2, C)$ as the limit of Y-Map sums. In order to define such maps we prove the Y-Map sums convergence. The discussion section 5 concludes the paper.

2 The Principal Series Matrix Coefficients Convergence

In this section we are going to prove two Lemmas stating that the following two sums are convergent: $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,\tau j)}(g)$ and $\sum_{j=m}^{\infty} D_{jm,jm}^{(j,\tau j)}(g)$ for a fixed m , where $g \in SL(2, C)$, $\tau \in C, j, n, m \in Z$. We will use these Lemmas in the following section in order to prove that the first sum converges to the non square integrable function on $SL(2, C)$, while the second sum converges to the square integrable function on $SL(2, C)$. Therefore the analog of the Peter-Weyl expansion can be derived for the second sum, while it does not exists for the first.

Lemma 1: The sum $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,\tau j)}(g)$ is absolute convergent and therefore

convergent for all $g \in SL(2, C)$, $\tau \in C$, $j, n, m \in Z$.

Proof:

According to the D'Alembert ratio test we need to prove:

$$\lim_{j \rightarrow \infty} \left| \frac{\sum_{|m| \leq j+1} \sum_{|n| \leq j+1} D_{(j+1)m, (j+1)n}^{((j+1)\tau(j+1))}(g)}{\sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm, jn}^{(j, \tau j)}(g)} \right| < 1 \quad (1)$$

Let us use the explicit expression for the matrix coefficients in (1). The first explicit expression of the principal series matrix coefficients $D_{jn, j'm}^{(k, \rho)}$, $k \in Z$, $\rho \in C$ was obtained by Duc and Hieu in 1967 [1], formula (4.11):

$$\begin{aligned} D_{jm, j'n}^{(k, \rho)}(g) &= \frac{\delta_{mn}}{(j + j' + 1)!} \\ &\times \sum_{d, d'} (-1)^{d+d'} \frac{(d + d' + m + k)!(j + j' - d - d' - m - k)!}{d!d'!(j - m - d)!(j' - m - d')!(k + m + d)!(k + m + d')!(j - k - d)!(j' - k - d')!} \\ &\times \epsilon^{2(2d' + m + k + 1 + \frac{i\rho}{2})} {}_2F_1(j' + 1 + \frac{i\rho}{2}, d + d' + m + k + 1; j + j' + 2; 1 - \epsilon^4) \quad (2) \end{aligned}$$

,where ${}_2F_1(\alpha, \beta; \gamma; z)$ - is a hypergeometric function, d and d' are integers that do not make each factor under the factorial to become a negative number and ϵ is a real number obtained from the $g \in SL(2, C)$ decomposition:

$$g = u_1 b u_2 \quad (3)$$

,where u_1 and u_2 are unitary matrices, while the matrix $b = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}$, $\epsilon \in R$

As one can see all $D_{jm, j'n}^{(k, \rho)}(g)$ are zero for $m \neq n$ due to the presence of the Kronecker delta in (2). Therefore we can omit all zero terms in the sums and leave only the terms with $m = n$. Thus, our sum becomes:

$$\sum_{j=0}^{\infty} \sum_{|m| \leq j} D_{jm, jm}^{(j, \tau j)}(g) \quad (4)$$

The matrix coefficients in our sum $D_{jm, jm}^{(j, \tau j)}(g)$ have much simpler form than the general form (2). We rewrite the unitary matrix coefficients $D_{jm, j'n}^{(k, \rho)}(g)$ in (2) for: $k = j$, $\rho = \tau j$, $j' = j$, $m = n$. Also since d and d' are so that factorial expressions are non-negative, one can see from (2) that if $k = j$, which is our case, then $j - k - d \geq 0$ implies $j - j - d \geq 0$, so $d \leq 0$, but at the same time $d!$ implies $d \geq 0$ so it follows

that $d = 0$. The same is true for $d' = 0$ and the sums over d and d' in (2) disappear:

$$D_{jm,jm}^{(j,\tau j)}(g) = \frac{1}{(2j+1)!} (2j)!(2j+1)(j+m)!(j-m)! \times \frac{(j+m)!(j-m)!}{(j-m)!(j-m)!(j+m)!(j+m)!} \\ \times \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \quad (5)$$

All coefficients cancel as one can see and we obtain:

$$D_{jm,jm}^{(j,\tau j)}(g) = \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \quad (6)$$

The sum (4) becomes:

$$\sum_{j=0}^{\infty} \sum_{|m| \leq j} D_{jm,jm}^{(j,\tau j)}(g) = \sum_{j=0}^{\infty} \sum_{|m| \leq j} \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \quad (7)$$

We now consider the following two sums: first for $0 \leq m \leq j$ and the second for $-j \leq m < 0$ and by bounding them from above we will prove their convergence. The convergence of the original sum will then follow.

$$\begin{aligned} \sum_{j=0}^{\infty} \left| \sum_{|m| \leq j} D_{jm,jm}^{(j,\tau j)}(g) \right| &\leq \\ \sum_{j=0}^{\infty} \left| \sum_{m=0}^m \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \right| &+ \\ \sum_{j=0}^{\infty} \left| \sum_{m=-j}^{m<0} \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \right| &\leq \\ \sum_{j=0}^{\infty} \sum_{m=0}^m \left| \epsilon^{2(m+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \right| &+ \\ \sum_{j=0}^{\infty} \sum_{m=-j}^{m<0} \left| \epsilon^{2(m+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) \right| &\leq \\ \sum_{j=0}^{\infty} \left| (j+1) \epsilon^{2(j+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, j+j+1; 2j+2; 1-\epsilon^4) \right| &+ \\ \sum_{j=0}^{\infty} \left| j \epsilon^{2(0+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, 0+j+1; 2j+2; 1-\epsilon^4) \right| &\quad (8) \end{aligned}$$

We pass to the last inequality above by putting $m = j$ in the first sum and $m = 0$ in the second and remembering the hypergeometric function is monotonic with respect to its second argument:

$${}_2F_1(a, b; c; z) = \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad (9)$$

,where

$$(q)_n = 1, \text{ when } n = 0, (q)_n = q(q+1)\dots(q+n-1), n > 0 \quad (10)$$

The hypergeometric function is originally defined for $|z| < 1$, but is analytically continued to all values of z as was shown in [5].

In our case of ${}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4)$, the parameter $b = m+j+1$ is always positive and the absolute value of the function is increasing when m is increasing. That is why in the last inequality of (8) we put $m = j$ to bound the sum from above when $m \geq 0$ and by $m = 0$ in the second sum, when $m < 0$.

At this point we are going to use the D'Alembert ratio convergence test and the asymptotic of the hypergeometric function to prove that the two bounding from above sums are convergent and that will prove that the original sum is convergent. We will need to consider three cases: $|\epsilon| < 1$, $|\epsilon| > 1$, $\epsilon = 1$

The hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; y)$ asymptotic, when all three parameters go to infinity, was investigated and derived by G.N Watson (1918) and can be found in Bateman's book [4] volume 1 page 77:

$$\begin{aligned} & \left(\frac{z}{2} - \frac{1}{2}\right)^{-a-\lambda} {}_2F_1(a+\lambda, a-c+1+\lambda; a-b+1+2\lambda; 2(1-z)^{-1}) = \\ & \frac{2^{a+b}\Gamma(a-b+1+2\lambda)\Gamma(1/2)\lambda^{-1/2}}{\Gamma(a-c+1+\lambda)\Gamma(c-b+\lambda)} e^{-(a+\lambda)\xi} \times (1-e^{-\xi})^{-c+1/2} \times (1+e^{-\xi})^{c-a-b-1/2} [1+O(\lambda^{-1})] \end{aligned} \quad (11)$$

,where ξ is defined as following: $e^{\pm\xi} = z \pm \sqrt{z^2 - 1}$. The minus sign corresponds to $Im(z) \leq 0$, the plus sign to $Im(z) > 0$. This asymptotic also works in the limit case of z being real, which is our case of $1-\epsilon^4$ (for details see Watson's original 1918 paper [5])

By comparing (6) and (11) we see that the hypergeometric function arguments λ, a, b, c in our case take the following values:

$$\lambda = j, a = 1 + \frac{i\tau j}{2}, b = \frac{i\tau j}{2}, c = 1 + \frac{i\tau j}{2} - m, z = \frac{\epsilon^4 + 1}{\epsilon^4 - 1}, e^{\mp\xi} = \frac{\epsilon^2 \mp 1}{\epsilon^2 \pm 1} \quad (12)$$

Indeed by substituting them into l.h.s of the (11) we get ${}_2F_1$ exactly as in (6):

$${}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4)$$

Let us rewrite (11) then in terms of (j, m, τ) and we obtain:

$$\begin{aligned} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) = \\ \frac{1}{(\epsilon^4-1)^{1+j+\frac{i\tau j}{2}}} \frac{2^{(1+i\tau j)}\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}}{\Gamma(m+1+j)\Gamma(1-m+j)} \times \\ e^{-(1+\frac{i\tau j}{2}+j)\xi} \times (1-e^{-\xi})^{(-\frac{1}{2}-\frac{i\tau j}{2}+m)} \times (1+e^{-\xi})^{(-m-\frac{i\tau j}{2}-\frac{1}{2})} \left[1+O(\frac{1}{j})\right] \end{aligned} \quad (13)$$

or by expressing $e^{-\xi}$ in terms of ϵ by using (12) we obtain the following expression:

$$\begin{aligned} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) = \\ \frac{1}{(\epsilon^4-1)^{1+j+\frac{i\tau j}{2}}} \frac{2^{(1+i\tau j)}\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}}{\Gamma(m+1+j)\Gamma(1-m+j)} \times \\ \left(\frac{\epsilon^2-1}{\epsilon^2+1}\right)^{(1+\frac{i\tau j}{2}+j)} \times \left(\frac{2}{\epsilon^2+1}\right)^{(-\frac{1}{2}-\frac{i\tau j}{2}+m)} \times \left(\frac{2\epsilon^2}{\epsilon^2+1}\right)^{(-m-\frac{i\tau j}{2}-\frac{1}{2})} \left[1+O(\frac{1}{j})\right] \end{aligned} \quad (14)$$

We are going to use this expression in the D'Alembert ratio test to prove the convergence of the bounding sums in (8). The first sum corresponds to $m = j$

$$\sum_{j=0}^{\infty} \left| (j+1)\epsilon^{2(j+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, j+j+1; 2j+2; 1-\epsilon^4) \right| \quad (15)$$

while the second to $m = 0$:

$$\sum_{j=0}^{\infty} \left| j\epsilon^{2(0+j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, 0+j+1; 2j+2; 1-\epsilon^4) \right| \quad (16)$$

By proving the sums convergence we would need to consider two cases of $|\epsilon| > 1$ and $|\epsilon| < 1$ for each sum separately, i.e. four cases all together. The simple fifth case $\epsilon = 1$ is considered at the end.

Case 1: First sum, $m = j$, $\tau \in C$, $\tau = \eta + i\omega$, $|\epsilon| > 1$

$$\sum_{j=0}^{\infty} \left| (j+1)\epsilon^{2(2j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4) \right| \quad (17)$$

The D'Alembert ratio test is as follows:

$$\lim_{j \rightarrow \infty} \left| \frac{(j+2)\epsilon^{2(2(j+1)+1+\frac{i\tau(j+1)}{2})}}{(j+1)\epsilon^{2(2j+1+\frac{i\tau j}{2})}} \right| \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, 2(j+1)+1; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4)} \right| =$$

$$\epsilon^4 \epsilon^{-\omega} \times \lim_{j \rightarrow \infty} \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, 2(j+1)+1; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4)} \right| \quad (18)$$

by using the Watson's asymptotic (14) for $m = j$ we obtain:

$$\lim_{j \rightarrow \infty} \epsilon^4 \epsilon^{-\omega} \times \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, 2(j+1)+1; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4)} \right| =$$

$$\lim_{j \rightarrow \infty} \epsilon^4 \epsilon^{-\omega} \times \left| \frac{(\epsilon^4 - 1)^{1+j+\frac{i\tau j}{2}}}{(\epsilon^4 - 1)^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1))\Gamma(\frac{1}{2})(j+1)^{-1/2}\Gamma(2j+1)\Gamma(1)}{\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}\Gamma(2(j+1)+1)\Gamma(1)} \right| \times$$

$$\left| \left(\frac{\epsilon^2 - 1}{\epsilon^2 + 1} \right)^{(1+\frac{i\tau(j+1)}{2}+(j+1))-(1+\frac{i\tau j}{2}+j)} \right| \times \left| \left(\frac{2}{\epsilon^2 + 1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2}+(j+1))-(-\frac{1}{2}-\frac{i\tau j}{2}+j)} \right|$$

$$\times \left| \left(\frac{2\epsilon^2}{\epsilon^2 + 1} \right)^{(-j-1-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-j-\frac{i\tau j}{2}-\frac{1}{2})} \right| =$$

$$\lim_{j \rightarrow \infty} \left| \frac{\epsilon^4 \epsilon^{-\omega} 2^{-\omega}}{(\epsilon^4 - 1)^{(1-\frac{\omega}{2})}} \frac{(2j+3)(2j+2)\Gamma(2j+2)\Gamma(2j+1)}{(2j+2)(2j+1)\Gamma(2j+2)\Gamma(2j+1)} \frac{(\epsilon^2 - 1)^{(1-\frac{\omega}{2})}}{(\epsilon^2 + 1)^{(1-\frac{\omega}{2})}} \frac{2^{(1+\frac{\omega}{2})}}{(\epsilon^2 + 1)^{(1+\frac{\omega}{2})}} \frac{(\epsilon^2 + 1)^{(1-\frac{\omega}{2})}}{(2\epsilon^2)^{(1-\frac{\omega}{2})}} \right| =$$

$$\frac{\epsilon^2}{(\epsilon^2 + 1)^2} < 1, \forall |\epsilon| > 1 \quad (19)$$

We used the fact that the absolute value of the positive real number in the pure imaginary power is 1 and the property of the Γ function: $\Gamma(z+1) = z\Gamma(z)$. By this property all Γ above cancel. We also remind that ω in the formula above comes from $\tau = \eta + i\omega$.

Case 2: First sum $m = j$, $\tau \in C$, $\tau = \eta + i\omega$, $|\epsilon| < 1$

$$\sum_{j=0}^{\infty} \left| (j+1)\epsilon^{2(2j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4) \right| \quad (20)$$

The D'Alembert ratio test provides the expression very similar to the Case 1 with one difference. In this case of $|\epsilon| < 1$ we write the following expressions in the form:

$$\epsilon^4 - 1 = (1 - \epsilon^4)e^{\pm i\pi} \quad (21)$$

$$\epsilon^2 - 1 = (1 - \epsilon^2)e^{\pm i\pi} \quad (22)$$

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \epsilon^4 \epsilon^{-\omega} \times \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, 2(j+1)+1; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, 2j+1; 2j+2; 1-\epsilon^4)} \right| = \\
& \lim_{j \rightarrow \infty} \epsilon^4 \epsilon^{-\omega} \times \left| \frac{((1-\epsilon^4)e^{\pm i\pi})^{1+j+\frac{i\tau j}{2}}}{((1-\epsilon^4)e^{\pm i\pi})^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1))\Gamma(\frac{1}{2})(j+1)^{-1/2}\Gamma(2j+1)\Gamma(1)}{\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}\Gamma(2(j+1)+1)\Gamma(1)} \right| \times \\
& \left| \left(\frac{(1-\epsilon^2)e^{\pm i\pi}}{\epsilon^2+1} \right)^{(1+\frac{i\tau(j+1)}{2}+(j+1))-(1+\frac{i\tau j}{2}+j)} \right| \times \left| \left(\frac{2}{\epsilon^2+1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2}+(j+1))-(\frac{1}{2}-\frac{i\tau j}{2}+j)} \right| \times \\
& \left| \left(\frac{2\epsilon^2}{\epsilon^2+1} \right)^{(-j-1-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-j-\frac{i\tau j}{2}-\frac{1}{2})} \right| = \\
& \lim_{j \rightarrow \infty} \left| \frac{\epsilon^4 \epsilon^{-\omega} 2^{-\omega}}{(1-\epsilon^4)^{(1-\frac{\omega}{2})} e^{\mp \frac{\pi\eta}{2}}} \frac{(2j+3)(2j+2)\Gamma(2j+2)\Gamma(2j+1)}{(2j+2)(2j+1)\Gamma(2j+2)\Gamma(2j+1)} \frac{(1-\epsilon^2)^{(1-\frac{\omega}{2})} e^{\mp \frac{\pi\eta}{2}}}{(\epsilon^2+1)^{(1-\frac{\omega}{2})}} \frac{2^{(1+\frac{\omega}{2})}}{(\epsilon^2+1)^{(1+\frac{\omega}{2})}} \frac{(\epsilon^2+1)^{(1-\frac{\omega}{2})}}{(2\epsilon^2)^{(1-\frac{\omega}{2})}} \right| = \\
& \frac{\epsilon^2}{(\epsilon^2+1)^2} < 1, \quad \forall |\epsilon| < 1 \quad (23)
\end{aligned}$$

Case 3: Second sum, $m = 0$, $\tau \in C$, $\tau = \eta + i\omega$, $|\epsilon| > 1$

$$\sum_{j=0}^{\infty} \left| j \epsilon^{2(j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1+\frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4) \right| \quad (24)$$

D'Alembert ratio test is as follows:

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \left| \frac{(j+1)\epsilon^{2(j+1+1+\frac{i\tau(j+1)}{2})}}{j\epsilon^{2(j+1+\frac{i\tau j}{2})}} \right| \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| = \\
& \epsilon^2 \epsilon^{-\omega} \times \lim_{j \rightarrow \infty} \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| \quad (25)
\end{aligned}$$

We use Watson's asymptotic (14) for $m = 0$

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} \left| \frac{{}_2F_1(j+2 + \frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1 + \frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| = \\
& \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} \times \left| \frac{(\epsilon^4 - 1)^{1+j+\frac{i\tau j}{2}}}{(\epsilon^4 - 1)^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1))\Gamma(\frac{1}{2})(j+1)^{-1/2}\Gamma(j+1)\Gamma(j+1)}{\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}\Gamma(j+2)\Gamma(j+2)} \right| \times \\
& \left| \left(\frac{\epsilon^2 - 1}{\epsilon^2 + 1} \right)^{(1+\frac{i\tau(j+1)}{2}+(j+1))-(1+\frac{i\tau j}{2}+j)} \right| \times \left| \left(\frac{2}{\epsilon^2 + 1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2})-(-\frac{1}{2}-\frac{i\tau j}{2})} \right| \\
& \times \left| \left(\frac{2\epsilon^2}{\epsilon^2 + 1} \right)^{(-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-\frac{i\tau j}{2}-\frac{1}{2})} \right| = \\
& \lim_{j \rightarrow \infty} \left| \frac{\epsilon^2 \epsilon^{-\omega} 2^{-\omega}}{(\epsilon^4 - 1)^{(1-\frac{\omega}{2})}} \frac{(2j+3)(2j+2)\Gamma(2j+2)\Gamma(j+1)\Gamma(j+1)}{(j+1)(j+1)\Gamma(2j+2)\Gamma(j+1)\Gamma(j+1)} \frac{(\epsilon^2 - 1)^{(1-\frac{\omega}{2})}}{(\epsilon^2 + 1)^{(1-\frac{\omega}{2})}} \frac{2^{\frac{\omega}{2}}}{(\epsilon^2 + 1)^{\frac{\omega}{2}}} \frac{(2\epsilon^2)^{\frac{\omega}{2}}}{(\epsilon^2 + 1)^{\frac{\omega}{2}}} \right| = \\
& \frac{4\epsilon^2}{(\epsilon^2 + 1)^2} < 1, \quad \forall |\epsilon| > 1 \quad (26)
\end{aligned}$$

Case 4: Second sum, $m = 0, \tau \in C, \tau = \eta + i\omega, |\epsilon| < 1$

$$\sum_{j=0}^{\infty} \left| j \epsilon^{2(j+1+\frac{i\tau j}{2})} \right| \left| {}_2F_1(j+1 + \frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4) \right| \quad (27)$$

This case is similar to Case 3. For $|\epsilon| < 1$ we need to write again the following two expressions in the form:

$$\epsilon^4 - 1 = (1 - \epsilon^4) e^{\pm i\pi} \quad (28)$$

$$\epsilon^2 - 1 = (1 - \epsilon^2) e^{\pm i\pi} \quad (29)$$

The D'Alembert ratio test is as follows:

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \left| \frac{(j+1)\epsilon^{2(j+1+\frac{i\tau(j+1)}{2})}}{j\epsilon^{2(j+1+\frac{i\tau j}{2})}} \right| \left| \frac{{}_2F_1(j+2 + \frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1 + \frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| = \\
& \epsilon^2 \epsilon^{-\omega} \times \lim_{j \rightarrow \infty} \left| \frac{{}_2F_1(j+2 + \frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1 + \frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| \quad (30)
\end{aligned}$$

By using the Watson's asymptotic (14) for $m = 0$ and $|\epsilon| < 1$

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} \left| \frac{{}_2F_1(j+2+\frac{i\tau(j+1)}{2}, j+2; 2(j+1)+2; 1-\epsilon^4)}{{}_2F_1(j+1+\frac{i\tau j}{2}, j+1; 2j+2; 1-\epsilon^4)} \right| = \\
& \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} \times \left| \frac{((1-\epsilon^4)e^{\pm i\pi})^{1+j+\frac{i\tau j}{2}}}{((1-\epsilon^4)e^{\pm i\pi})^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1))\Gamma(\frac{1}{2})(j+1)^{-1/2}\Gamma(j+1)\Gamma(j+1)}{\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}\Gamma(j+2)\Gamma(j+2)} \right| \times \\
& \left| \left(\frac{(1-\epsilon^2)e^{\pm i\pi}}{\epsilon^2+1} \right)^{(1+\frac{i\tau(j+1)}{2}+(j+1))-(1+\frac{i\tau j}{2}+j)} \right| \times \left| \left(\frac{2}{\epsilon^2+1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2})-(-\frac{1}{2}-\frac{i\tau j}{2})} \right| \\
& \times \left| \left(\frac{2\epsilon^2}{\epsilon^2+1} \right)^{(-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-\frac{i\tau j}{2}-\frac{1}{2})} \right| = \\
& \lim_{j \rightarrow \infty} \left| \frac{\epsilon^2 \epsilon^{-\omega} 2^{-\omega}}{(1-\epsilon^4)^{(1-\frac{\omega}{2})} e^{\mp \frac{\pi\eta}{2}}} \frac{(2j+3)(2j+2)\Gamma(2j+2)\Gamma(j+1)\Gamma(j+1)}{(j+1)(j+1)\Gamma(2j+2)\Gamma(j+1)\Gamma(j+1)} \frac{(1-\epsilon^2)^{(1-\frac{\omega}{2})} e^{\mp \frac{\pi\eta}{2}}}{(\epsilon^2+1)^{(1-\frac{\omega}{2})}} \frac{2^{\frac{\omega}{2}}}{(\epsilon^2+1)^{\frac{\omega}{2}}} \frac{(2\epsilon^2)^{\frac{\omega}{2}}}{(\epsilon^2+1)^{\frac{\omega}{2}}} \right| = \\
& \frac{4\epsilon^2}{(\epsilon^2+1)^2} < 1, \quad \forall |\epsilon| < 1 \quad (31)
\end{aligned}$$

Case 5: $\epsilon = 1$

The remaining case $\epsilon = 1$ is trivial as ${}_2F_1(a, b; c; 0) = 0$, which follows from the hypergeometric function definition (9).

We have proved the D'Alembert ratio test for all five cases. Thus it follows that the bounding sums (15) and (16) are absolute convergent and therefore by (8) the sum $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm, jn}^{(j, \tau j)}(g)$ is absolute convergent and therefore convergent for all $g \in SL(2, C)$, $\tau \in C$, $j, n, m \in Z$.

It is also clear by construction that the sum is convergent to the function on $SL(2, C)$. To every $g \in SL(2, C)$ there is a corresponding complex number, that is the sum limit. Since the sum limit is unique for each g by construction, the sum is convergent to the function on $SL(2, C)$.

□

Lemma 2: The sum $\sum_{j=0}^{\infty} D_{jm, jm}^{(j, \tau j)}(g)$ is absolute convergent and therefore convergent for each $m \in Z$, $\forall g \in SL(2, C)$, $\tau \in C$, $j \in Z$.

Proof:

The proof is very similar to the proof of the Lemma 1. We need to consider only three cases $|\epsilon| > 1$, $|\epsilon| < 1$ and $|\epsilon| = 1$, instead of five cases of the Lemma 1. This is due to the absence of the sums over m and n and therefore there is no need in the bounding

sums.

According to the D'Alembert ratio test we need to prove:

$$\lim_{j \rightarrow \infty} \left| \frac{D_{(j+1)m, (j+1)m}^{((j+1), \tau(j+1))}(g)}{D_{jm, jm}^{(j, \tau j)}(g)} \right| < 1 \quad (32)$$

By using the explicit form of the matrix coefficients (6) and the asymptotic (14) we can write the asymptotic of the matrix coefficients in the form:

$$\begin{aligned} D_{jm, jm}^{(j, \tau j)}(g) &= \epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4) = \\ &= \frac{1}{(\epsilon^4-1)^{1+j+\frac{i\tau j}{2}}} \frac{2^{(1+i\tau j)} \Gamma(2+2j) \Gamma(\frac{1}{2}) j^{-1/2}}{\Gamma(m+1+j) \Gamma(1-m+j)} \times \\ &= \left(\frac{\epsilon^2-1}{\epsilon^2+1} \right)^{(1+\frac{i\tau j}{2}+j)} \times \left(\frac{2}{\epsilon^2+1} \right)^{(-\frac{1}{2}-\frac{i\tau j}{2}+m)} \times \left(\frac{2\epsilon^2}{\epsilon^2+1} \right)^{(-m-\frac{i\tau j}{2}-\frac{1}{2})} \left[1 + O\left(\frac{1}{j}\right) \right] \end{aligned} \quad (33)$$

We substitute this expression into (32) and consider three cases $|\epsilon| > 1$, $|\epsilon| < 1$ and $|\epsilon| = 1$:

Case 1: $|\epsilon| > 1$, $\tau \in C$, $\tau = \eta + i\omega$

$$\begin{aligned} \lim_{j \rightarrow \infty} \left| \frac{D_{(j+1)m, (j+1)m}^{((j+1), \tau(j+1))}(g)}{D_{jm, jm}^{(j, \tau j)}(g)} \right| &= \left| \frac{\epsilon^{2(m+j+2+\frac{i\tau(j+1)}{2})} {}_2F_1(j+2+\frac{i\tau(j+1)}{2}, m+j+2; 2j+4; 1-\epsilon^4)}{\epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4)} \right| = \\ \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} \times &\left| \frac{(\epsilon^4-1)^{1+j+\frac{i\tau j}{2}}}{(\epsilon^4-1)^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1)) \Gamma(\frac{1}{2}) (j+1)^{-1/2} \Gamma(m+1+j) \Gamma(1-m+j)}{\Gamma(2+2j) \Gamma(\frac{1}{2}) j^{-1/2} \Gamma(m+2+j) \Gamma(2-m+j)} \right| \times \\ &\left| \left(\frac{\epsilon^2-1}{\epsilon^2+1} \right)^{(1+\frac{i\tau(j+1)}{2}+j+1)-(1+\frac{i\tau j}{2}+j)} \right| \times \left| \left(\frac{2}{\epsilon^2+1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2}+m)-(-\frac{1}{2}-\frac{i\tau j}{2}+m)} \right| \\ &\times \left| \left(\frac{2\epsilon^2}{\epsilon^2+1} \right)^{(-m-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-m-\frac{i\tau j}{2}-\frac{1}{2})} \right| = \\ \lim_{j \rightarrow \infty} \left| \frac{\epsilon^2 \epsilon^{-\omega} 2^{-\omega}}{(\epsilon^4-1)^{(1-\frac{\omega}{2})}} \frac{(2j+3)(2j+2) \Gamma(2j+2) \Gamma(m+1+j) \Gamma(1-m+j)}{\Gamma(2j+2) (m+1+j) (1-m+j) \Gamma(m+1+j) \Gamma(1-m+j)} \right| \times \\ &\left| \frac{(\epsilon^2-1)^{(1-\frac{\omega}{2})}}{(\epsilon^2+1)^{(1-\frac{\omega}{2})}} \frac{2^{\frac{\omega}{2}}}{(\epsilon^2+1)^{\frac{\omega}{2}}} \frac{(2\epsilon^2)^{(\frac{\omega}{2})}}{(\epsilon^2+1)^{(\frac{\omega}{2})}} \right| = \frac{4\epsilon^2}{(\epsilon^2+1)^2} < 1, \quad \forall |\epsilon| > 1 \quad (34) \end{aligned}$$

The Case 2 is very similar to the Case 1; the only difference is that, when $|\epsilon| < 1$ we need to write the following expressions in the form:

$$\epsilon^4 - 1 = (1 - \epsilon^4) e^{\pm i\pi} \quad (35)$$

$$\epsilon^2 - 1 = (1 - \epsilon^2)e^{\pm i\pi} \quad (36)$$

Case 2: $|\epsilon| < 1, \tau \in C, \tau = \eta + i\omega$

$$\begin{aligned} \lim_{j \rightarrow \infty} \left| \frac{D_{(j+1)m, (j+1)m}^{((j+1), \tau(j+1))}(g)}{D_{jm, jm}^{(j, \tau j)}(g)} \right| &= \left| \frac{\epsilon^{2(m+j+2+\frac{i\tau(j+1)}{2})} {}_2F_1(j+2+\frac{i\tau(j+1)}{2}, m+j+2; 2j+4; 1-\epsilon^4)}{\epsilon^{2(m+j+1+\frac{i\tau j}{2})} {}_2F_1(j+1+\frac{i\tau j}{2}, m+j+1; 2j+2; 1-\epsilon^4)} \right| = \\ \lim_{j \rightarrow \infty} \epsilon^2 \epsilon^{-\omega} &\left| \frac{((1-\epsilon^4)e^{\pm i\pi})^{1+j+\frac{i\tau j}{2}}}{((1-\epsilon^4)e^{\pm i\pi})^{1+j+1+\frac{i\tau(j+1)}{2}}} \frac{2^{(1+i\tau(j+1))}}{2^{(1+i\tau j)}} \frac{\Gamma(2+2(j+1))\Gamma(\frac{1}{2})(j+1)^{-1/2}\Gamma(m+1+j)\Gamma(1-m+j)}{\Gamma(2+2j)\Gamma(\frac{1}{2})j^{-1/2}\Gamma(m+2+j)\Gamma(2-m+j)} \right| \\ \times \left| \left(\frac{(1-\epsilon^2)e^{\pm i\pi}}{\epsilon^2+1} \right)^{(1+\frac{i\tau(j+1)}{2}+j+1)-(1+\frac{i\tau j}{2}+j)} \right| &\times \left| \left(\frac{2}{\epsilon^2+1} \right)^{(-\frac{1}{2}-\frac{i\tau(j+1)}{2}+m)-(-\frac{1}{2}-\frac{i\tau j}{2}+m)} \right| \\ \times \left| \left(\frac{2\epsilon^2}{\epsilon^2+1} \right)^{(-m-\frac{i\tau(j+1)}{2}-\frac{1}{2})-(-m-\frac{i\tau j}{2}-\frac{1}{2})} \right| &= \\ \lim_{j \rightarrow \infty} \left| \frac{\epsilon^2 \epsilon^{-\omega} 2^{-\omega}}{((1-\epsilon^4)e^{\pm i\pi})^{(1-\frac{\omega}{2})}} \frac{(2j+3)(2j+2)\Gamma(2j+2)\Gamma(m+1+j)\Gamma(1-m+j)}{\Gamma(2j+2)(m+1+j)(1-m+j)\Gamma(m+1+j)\Gamma(1-m+j)} \right| \times \\ \left| \frac{((1-\epsilon^2)e^{\pm i\pi})^{(1-\frac{\omega}{2})}}{(\epsilon^2+1)^{(1-\frac{\omega}{2})}} \frac{2^{\frac{\omega}{2}}}{(\epsilon^2+1)^{\frac{\omega}{2}}} \frac{(2\epsilon^2)^{(\frac{\omega}{2})}}{(\epsilon^2+1)^{(\frac{\omega}{2})}} \right| &= \frac{4\epsilon^2}{(\epsilon^2+1)^2} < 1, \forall |\epsilon| < 1 \quad (37) \end{aligned}$$

Case 3: $|\epsilon| = 1, \tau \in C, \tau = \eta + i\omega$

The sum is convergent and equals to zero in this case since ${}_2F_1(a, b; c, 1 - \epsilon^4) = {}_2F_1(a, b; c, 0) = 0$.

This completes the proof of the sum absolute and therefore regular convergence.

By the same argument as at the end of the Lemma 1 it is clear that by construction the limit of the sum is the function on $SL(2, C)$.

□

3 Square Integrability

Theorem 1 The limit of the sum $\sum_{j=0}^{\infty} D_{jm, jm}^{(j, \tau j)}(g), \tau \in C, j, m \in Z$ is a square integrable function for all $\tau \neq i, -i$ and fixed m .

Proof:

Let $f(g)$ be the sum limit. Consider the inner product integral:

$$\langle f(g), \overline{f(g)} \rangle = \int_{SL(2, C)} f(g) \overline{f(g)} dg = \int_{SL(2, C)} dg \left(\sum_{j=0}^{\infty} D_{jm, jm}^{j, \tau j}(g) \right) \left(\sum_{j'=0}^{\infty} \overline{D_{j'm', j'm'}^{j', \tau j'}(g)} \right) \quad (38)$$

The Lorentz matrix coefficients square integrability and orthogonality provides us the following equality [2], formula 9:

$$\int_{SL(2,C)} D_{j_1 q_1, j'_1 q'_1}^{(n_1, \rho_1)}(g) \overline{D_{j_2 q_2, j'_2 q'_2}^{(n_2, \rho_2)}(g)} dg = \delta_{n_1 n_2} \frac{\delta(\rho_1 - \rho_2)}{n_1^2 + \rho_1^2} \delta_{(j_1 q_1), (j_2 q_2)} \delta_{(j'_1 q'_1), (j'_2 q'_2)} \quad (39)$$

where, $n_1, n_2 \in \mathbb{Z}, \rho_1, \rho_2 \in \mathbb{C}$. From (38), and (39) we immediately obtain:

$$\langle f(g), \overline{f(g)} \rangle = \int_{SL(2,C)} f(g) \overline{f(g)} dg = \sum_{j=1}^{\infty} \frac{1}{j^2 + \tau^2 j^2} = \frac{\pi^2}{6} \frac{1}{\tau^2 + 1} \quad (40)$$

We used the fact that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, and we started the sum from $j = 1$ since for $j = 0$

$$D_{jm, jm}^{(j, j\tau)} = 0$$

The integral exists for all values $\tau \neq i, -i$.

□

Theorem 2 The limit of the sum $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm, jn}^{(j, \tau j)}(g), \tau \in \mathbb{C}$ is not square integrable.

Proof:

Let $f(g)$ be the sum limit. Consider the inner product integral:

$$\langle f(g), \overline{f(g)} \rangle = \int_{SL(2,C)} f(g) \overline{f(g)} dg = \int_{SL(2,C)} dg \left(\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm, jn}^{(j, \tau j)}(g) \right) \left(\sum_{j'=0}^{\infty} \sum_{|m'| \leq j'} \sum_{|n'| \leq j'} \overline{D_{j'm', j'n'}^{(j', \tau j')}(g)} \right) \quad (41)$$

By using matrix coefficients orthogonality we obtain:

$$\langle f(g), \overline{f(g)} \rangle = \int_{SL(2,C)} f(g) \overline{f(g)} dg = \sum_{j=1}^{\infty} \frac{2j+1}{j^2 + \tau^2 j^2} \quad (42)$$

we started the sum from $j = 1$ since for $j = 0$ $D_{jm, jm}^{(j, j\tau)} = 0$

The sum on the right is divergent therefore the function $f(g)$ is not square integrable.

□

Let us prove that any square integrable function on $SL(2, \mathbb{C})$ can be expanded into the following sum:

$$\psi(g) = \sum_{j=0}^{\infty} j^2 (1 + \tau^2) c_{jmm} D_{jm, jm}^{(j, \tau j)}(g) \quad (43)$$

,where $\tau \neq i, -i$, m is fixed and c_{jmm} are the Fourier coefficients:

$$c_{jmm} = \int_{SL(2,C)} \psi(g) \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg \quad (44)$$

Lemma 3 Let $\psi(g) \in L_2(SL(2, C))$, $g \in SL(2, C)$ be a square integrable function and c_{jmm} being its Fourier transform coefficients: $c_{jmm} = \int_{SL(2,C)} \psi(g) \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg$

then the sum $\sum_{j=0}^{\infty} c_{jmm}$ is convergent.

Proof:

$$\sum_{j=0}^{\infty} c_{jmm} = \sum_{j=0}^{\infty} \int_{SL(2,C)} \psi(g) \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg = \int_{SL(2,C)} \psi(g) \sum_{j=0}^{\infty} \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg \quad (45)$$

However by Theorem 1 the sum $\sum_{j=0}^{\infty} D_{jm,jm}^{(j,\tau j)}(g)$ is square integrable and therefore the

$\sum_{j=0}^{\infty} \overline{D_{jm,jm}^{(j,\tau j)}(g)}$ is also square integrable and converges to the square integrable function $\phi(g) \in L_2(SL(2, C))$. Therefore the integral $\int_{SL(2,C)} \psi(g) \phi(g) dg$ in (45) is conver-

gent and so the sum $\sum_{j=0}^{\infty} c_{jmm}$.

□

Lemma 4 The sum $\sum_{j=0}^{\infty} j^2(1 + \tau^2)c_{jmm}D_{jm,jm}^{(j,\tau j)}(g)$ is convergent for a fixed m .

Proof:

We again use the D'Alambert ratio test:

$$\lim_{j \rightarrow \infty} \left| \frac{(j+1)^2(1 + \tau^2)}{j^2(1 + \tau^2)} \right| \left| \frac{c_{(j+1)mm}}{c_{jmm}} \right| \left| \frac{D_{(j+1)m,(j+1)m}^{(j+1,\tau(j+1))}(g)}{D_{jm,jm}^{(j,\tau j)}(g)} \right| < 1 \quad (46)$$

The inequality is true since:

$$\lim_{j \rightarrow \infty} \left| \frac{(j+1)^2(1 + \tau^2)}{j^2(1 + \tau^2)} \right| = 1, \text{ and}$$

$$\lim_{j \rightarrow \infty} \left| \frac{c_{(j+1)mm}}{c_{jmm}} \right| \leq 1 \text{ due to Lemma 3, (45).}$$

$$\lim_{j \rightarrow \infty} \left| \frac{D_{(j+1)m,(j+1)m}^{(j+1,\tau(j+1))}(g)}{D_{jm,jm}^{(j,\tau j)}(g)} \right| = \frac{4\epsilon^2}{(\epsilon^2+1)^2} < 1 \text{ due to Lemma 2 (34), (37).}$$

□

Theorem 3 Any square intregable function $\psi(g) \in L_2(SL(2, C))$, $g \in SL(2, C)$

can be expanded into the following sum:

$$\psi(g) = \sum_{j=0}^{\infty} j^2 (1 + \tau^2) c_{jmm} D_{jm,jm}^{(j,\tau j)}(g) \quad (47)$$

,where $\tau \in C, \tau \neq i, -i, j \in Z, c_{jmm}$ are the Fourier coefficients:

$$c_{jmm} = \int_{SL(2,C)} \psi(g) \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg \quad (48)$$

Proof:

By Lemma 4 the sum $\sum_{j=0}^{\infty} j^2 (1 + \tau^2) c_{jmm} D_{jm,jm}^{(j,\tau j)}(g)$ is convergent to some function $f(g), g \in SL(2, C)$.

$$\sum_{j=0}^{\infty} j^2 (1 + \tau^2) c_{jmm} D_{jm,jm}^{(j,\tau j)}(g) = f(g) \quad (49)$$

By multiplying both sides of (49) by $\overline{D_{j'm,j'n}^{(j',\tau j')}(g)}$ and integrating we obtain:

$$\int_{SL(2,C)} f(g) \overline{D_{j'm,j'n}^{(j',\tau j')}(g)} dg = \sum_{j=0}^{\infty} j^2 (1 + \tau^2) c_{jmm} \int_{SL(2,C)} D_{jm,jm}^{(j,\tau j)}(g) \overline{D_{j'm,j'n}^{(j',\tau j')}(g)} dg \quad (50)$$

Finally by using the matrix coefficients orthogonality (39) we arrive at:

$$\int_{SL(2,C)} f(g) \overline{D_{jm,jm}^{(j,\tau j)}(g)} dg = c_{jmm} \quad (51)$$

The c_{jmm} definition (48) then implies that $\psi(g) = f(g)$

□

4 The Y-Map: $L_2(SU(2)) \rightarrow F(SL(2, C))$

In Lemma 1, we have proved that the sum $\sum_{j=0}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,\tau j)}(g), \tau \in C$ is convergent. In Theorem 2, we have also proved that its limit is not a square integrable function and therefore can not be used in the analog of the Peter-Weyl theorem for the Lorentz group. Thus we had to use another expansion. However the above sum can be used to define a map from the space of functions on $SU(2)$ to the space of functions on $SL(2, C)$. Such map was first introduced in the Loop Quantum Gravity [10], [11] and called the Y-Map. Below we correct the definition of the Y-Map and prove the Y-Map

sums convergence. We are going to prove that any square integrable function $\phi(u)$ on $SU(2)$ can be mapped to the function $\psi(g)$ on $SL(2, C)$ in the following manner:

$$\phi(u) \rightarrow \psi(g) = \sum_{j=|p|}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} d_{|p|m}^{\frac{j}{2}} D_{jm, jn}^{(j, \tau j)}(g) \quad (52)$$

where $d_{|p|m}^{\frac{j}{2}} = (j+1)^{\frac{1}{2}} \int_{SU(2)} \phi(u) \overline{D_{|p|m}^{\frac{j}{2}}(u)} du$

is $\phi(u)$'s Fourier transform and $p, j, m, n \in Z, \tau \in C, u \in SU(2), g \in SL(2, C)$

Please note that this definition is a little different from [10], [11] as the j parameter of the $SL(2, C)$ matrix coefficient is an integer, while the parameter of the $SU(2)$ in $d_{|p|m}^{\frac{j}{2}}$ is a half-integer $j/2$.

Let's prove that the sum (52) converges to a function on $SL(2, C)$

Lemma 5 The sum $\sum_{k=\frac{|p|}{2}}^{\infty} \sum_{|m| \leq k} d_{\frac{|p|}{2}m}^k$ is convergent, where $k = \frac{|p|}{2} + n, n \in N, p \in Z$

Proof:

By the Paley-Wiener Theorem ([12] page 60, 91, see also [18]) the Fourier transform $d_{\frac{|p|}{2}m}^k$ satisfies the following asymptotic inequality:

$$\lim_{k \rightarrow \infty} \sup_m (|d_{\frac{|p|}{2}m}^k k^n) = 0, \quad \forall n \in N \quad (53)$$

or

$$|k^n d_{\frac{|p|}{2}m}^k| \leq C_n \quad (54)$$

$\forall n \in N$ or we can rewrite it as:

$$|d_{\frac{|p|}{2}m}^k| \leq \frac{C_n}{|k|^n} \quad (55)$$

which means that the Fourier transform is a fast dropping function and decreases faster than any polynomial of power n . Then the sum :

$$\sum_{k=\frac{|p|}{2}}^{\infty} \sum_{|m| \leq k} |d_{\frac{|p|}{2}m}^k| \leq \sum_{k=\frac{|p|}{2}}^{\infty} \sum_{|m| \leq k} \frac{C_n}{|k|^n} \leq C_n \sum_{k=\frac{|p|}{2}}^{\infty} \frac{(2k+1)}{|k|^n} \quad (56)$$

and the latter is a Riemann zeta function and is convergent $\forall n > 2$.

This proves the absolute convergence and therefore the regular convergence.

If we pass in the notation from the half-integer k to the integer j by writing $k = \frac{j}{2}$, we

obtain that $\sum_{j=|p|}^{\infty} \sum_{|m| \leq j} d_{|p|m}^{\frac{j}{2}}$ is convergent.

□

Theorem 4 The sum $\sum_{j=|p|}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} d_{|p|m}^{\frac{j}{2}} D_{jm,jn}^{(j,\tau j)}(g)$ is convergent.

Proof:

$$\sum_{j=|p|}^k \sum_{|m| \leq j} \sum_{|n| \leq j} d_{|p|m}^{\frac{j}{2}} D_{jm,jn}^{(j,\tau j)}(g) \leq \sum_{j=|p|}^k \sum_{|m| \leq j} |d_{|p|m}^{\frac{j}{2}}| \times \sum_{j=0}^k \left| \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,\tau j)}(g) \right| \quad (57)$$

$\forall k \geq |p|$ and therefore it is true in the limit when $k \rightarrow \infty$.

$$\sum_{j=|p|}^{\infty} \sum_{|m| \leq j} \sum_{|n| \leq j} d_{|p|m}^{\frac{j}{2}} D_{jm,jn}^{(j,\tau j)}(g) \leq \sum_{j=|p|}^{\infty} \sum_{|m| \leq j} |d_{|p|m}^{\frac{j}{2}}| \times \sum_{j=0}^{\infty} \left| \sum_{|m| \leq j} \sum_{|n| \leq j} D_{jm,jn}^{(j,\tau j)}(g) \right| \quad (58)$$

The first sum on the right hand side converges due to the Lemma 5 above. The second sum converges due to the Lemma 1.

The limit is a function on $SL(2, C)$ since each $g \in SL(2, C)$ we map to the sum limit and the limit is unique by construction.

□

The Theorem 4 establishes the map from the space of the square integrable functions on $SU(2)$ to the space of the functions (not necessarily square integrable) on $SL(2, C)$.

5 Discussion

We have proved the analog of the Peter-Weyl expansion into to the discrete sum of the Lorentz group principal series matrix coefficients selected in a special manner. The basis consists of the matrix coefficients of the form $D_{jm,jm}^{(j,\tau j)}$, where $\tau \in C, \tau \neq i, -i$ and $j, m \in Z$. The expansion is quite different from the analog of the Plancherel formula for the Lorentz group [12]. While the analog of the Plancherel formula contains the sum over the principal series parameter k and the integral over the complex parameter ρ of the principal series matrix coefficients $D_{jm,j'n}^{(k,\rho)}$ and the sum is over all parameters, the new expansion contains only the sum and no integral and sum is over the selected diagonal matrix coefficients of the form $D_{jm,jm}^{(j,\tau j)}$. We proved the convergence of such sum and the square integrability of the limit of such expansion. We have also proved the convergence of the sums in the Y-Map from the space of the square integrable functions on $SU(2)$ to the space of functions on $SL(2, C)$.

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